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Local stability analysis of dynamic model on dengue transmission

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Abstract
Dengue fever transmission in Indonesia has an advanced amount. In this article, dynamic model of interaction between human and Aedes aegypti mosquitos is learned. The SEIRRD (Susceptible, Exposed, Infected, Recovered, Deceased) model is used in this article. The purpose in this model is to describe the stability of dengue transmission, so that we can analyze the developed of epidemic model in mathematic field. Using NGM method to analyze basic reproduction number and applying Routh-Hurwitz criteria method to show the local stability of model. Then, two equilibrium points, called endemic and non-endemic equilibrium points, are obtained. The result of basic reproduction number is described the stability analysis. If basic reproduction number less than one, the endemic equilibrium point is locally asymptotically stable and otherwise. Local stability analysis at the equilibrium point is determined through parameter analysis. Furthermore, numerical simulations are carried out by fitting the data to obtained the result of the parameters. The results of numerical simulations explain the spread of dengue transmission.

Keywords
SEIRRD model, Dengue transmission, Local stability analysis

Introduction
Dengue Fever is a disease that can be life-threatening. This disease is caused by infection by the dengue virus which is transmitted through the bite of the Aedes aegypti mosquitos [1]. Infected sufferers will have symptoms in the form of mild to high fever, accompanied by headaches, pain in the eyes, muscles and joints, and even spontaneous bleeding [2]. There are four different types of dengue virus, which can cause dengue fever. In Indonesia, dengue fever cases are still increasing, with almost 10% dying. Tropical and subtropical climate countries are at high risk of transmission of the virus, especially in the humid rainy season [3]. The world health organization estimates that every year there are 50-100 million cases of dengue virus infection throughout the world. Acute dengue fever is accompanied by bleeding manifestations which tend to result in shocks which can cause death [4], this disease occurs acutely and attacks both

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adults and children under 15 years of age [5]. So, this dengue case should immediately receive appropriate treatment so that the transmission of the dengue virus does not spread further. Therefore, this dengue case cannot be ignored, so we must know the pattern of spread of the dengue virus.

In mathematical modelling, various earlier research has axamined the dengue transmission model, some of which include the SIR-SI model was proposed by [6], then the SIR model is developed by [7], which include exposed variable for human become SEIR-SI model. Then by [8] adding exposed variable for mosquito become SEIR-SEI model.

In this study, the authors expand upon the SEIR-SEI model introduced by [8] by incorporating a variable for deceased individuals in the spread of dengue illness. Additionally, we conduct a stability analysis of the model to identify two equilibrium points: the endemic equilibrium (EE) and the disease-free equilibrium (DFE) point. We utilize the basic reproduction number ($\Re_0$) to assess the stability of the model and employ the Routh-Hurwitz criteria method to determine the local stability of the system. The value of $\Re_0$ obtained from our analysis is used to evaluate the stability of the model. Through numerical simulations, we illustrate the dynamics of dengue illness propagation, utilizing data from the sources cited in [8].

Methods

The method from this research begins with formulating a model, then we get the equilibrium point. From this, we analyze the existence and stability of equilibrium point, before that we calculate the basic reproduction number. We use Next Generation Matrix (NGM) and Routh Hurwitz Criteria method to do these steps. This is the steps from this research methodology.

Model Formulation

Based on the dengue virus transmission, we developed the SEIR (susceptible-exposed-infected-recovered) model by adding the deceased variable for human and we develodep the SI (susceptible-infected) by adding the exposed variable for mosquuitos. We divided the total human population into five subpopulations: susceptible human (S_h), exposed human (E_h), infected human (I_h), recovered human (R_h), deceased human (D_h) and then the total mosquito's population is divided into three subpopulations: susceptible mosquito (S_m), exposed mosquito (E_m), and infected mosquito (I_m).

For mathematical transformation of real situations, we make the following assumption:

1. Both humans and mosquitos are closed
2. Only infected mosquito and infected human can spread the disease [9]
3. The rate of mosquito's bite is b
4. The exposed humans and mosquitos become infected subpopulation with rate ρ
5. All of infected mosquitos become death

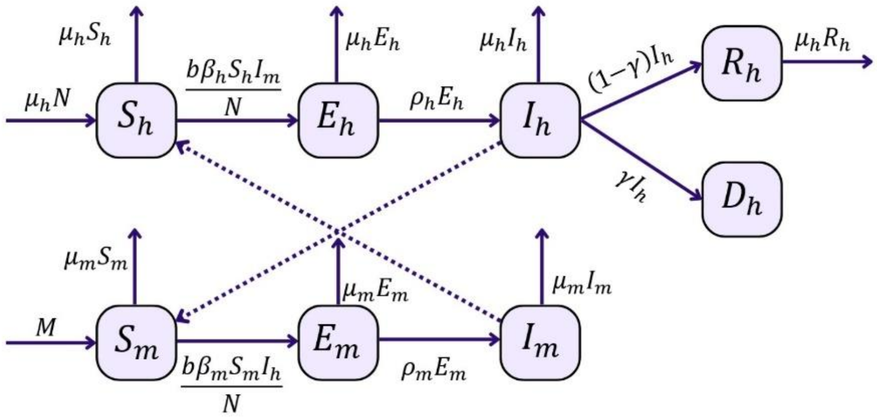

Figure 1. Diagram of SEIRD-SEI Dengue Model

Table 1. Descriptions of Parameters

Notation	Interpretations	Notation	Interpretations
μ_h	The natural death rate of human	μ_m	The natural death rate of mosquito
β_h	The mosquitos-human transmission probability	β_m	The human-mosquitos transmission probability
ρ_h	The human incubation rate	ρ_m	The mosquito incubation rate
γ_h	Rate of human infected become deceased	$1-\gamma_h$	Rate of human infected become recovered
b	The bitting rate of the mosquito		

The developed mathematical model as follows the model from [8]. The model from [8] give the SEIR model for human and SEI model for mosquito. We developed the human model by adding the deceased variable, so that the model become SEIRD-SEI model of dengue transmission. So the mathematical model is given as follows:

$$\begin{cases} \frac{dS_h}{dt} = \mu_h N - \frac{b\beta_h S_h I_m}{N} - \mu_h S_h \\ \frac{dE_h}{dt} = \frac{b\beta_h S_h I_m}{N} - \rho_h E_h - \mu_h E_h \\ \frac{dI_h}{dt} = \rho_h E_h - (\gamma + \mu_h) I_h - (1-\gamma) I_h \\ \frac{dR_h}{dt} = (1-\gamma) I_h - \mu_h R_h \\ \frac{dD_h}{dt} = \gamma_h I_h \\ \frac{dS_m}{dt} = M - \frac{b\beta_m S_m I_h}{N} - \mu_m S_m \\ \frac{dE_m}{dt} = \frac{b\beta_m S_m I_h}{N} - (\rho_m + \mu_m) E_m \\ \frac{dI_m}{dt} = \rho_m E_m + \mu_m I_m \end{cases} \tag{1}$$

With the initial condition $S_h(0) \geq 0, E_h(0) \geq 0, I_h(0) \geq 0, R_h(0) \geq 0, D_h(0) \geq 0, S_m(0) \geq 0, E_m(0) \geq 0, I_m(0) \geq 0$ where the parameters are given in Table 1.

Equilibrium Point of the Dynamical Model

The equilibrium point is a condition where there are no changes in each population over time. System (1) can be written:

$$\begin{cases} \frac{dS_h}{dt} = \mu_h N - \frac{b\beta_h S_h I_m}{N} - \mu_h S_h = 0 \\ \frac{dE_h}{dt} = \frac{b\beta_h S_h I_m}{N} - \rho_h E_h - \mu_h E_h = 0 \\ \frac{dI_h}{dt} = \rho_h E_h - (\gamma + \mu_h) I_h - (1-\gamma) I_h = 0 \\ \frac{dR_h}{dt} = (1-\gamma) I_h - \mu_h R_h = 0 \\ \frac{dD_h}{dt} = \gamma_h I_h = 0 \\ \frac{dS_m}{dt} = M - \frac{b\beta_m S_m I_h}{N} - \mu_m S_m = 0 \\ \frac{dE_m}{dt} = \frac{b\beta_m S_m I_h}{N} - (\rho_m + \mu_m) E_m = 0 \\ \frac{dI_m}{dt} = \rho_m E_m + \mu_m I_m = 0 \end{cases} \tag{2}$$

The system of equations (1) comprises two equilibrium points : the endemic equilibrium (EE), designated by $\mathcal{E}^*(S_h^*, E_h^*, I_h^*, R_h^*, S_m^*, E_m^*, I_m^*)$, and disease-free/non-endemic equilibrium (DFE), designated by $\mathcal{E}_0(S_{h_0}, E_{h_0}, I_{h_0}, R_{h_0}, S_{m_0}, E_{m_0}, I_{m_0}) = \left(N, 0,0,0, \frac{M}{\mu_m}, 0,0\right)$.

There is a situation known as the endemic equilibrium point where the dispersion of Dengue fever occurs so that $S_h, E_h, I_h, R_h, S_m, E_m, I_m$ variables are not equal to zero. Then with the aid of Maple, $\mathcal{E}^*(S_h^*, E_h^*, I_h^*, R_h^*, S_m^*, E_m^*, I_m^*)$, are obtained with the respective values as follows:

$$S_h^* = \frac{N^2 \mu_m E C}{\beta_m \rho_h b(M b \beta_h \rho_m + N \rho_m \mu_m \mu_h + N \mu_m^2 \mu_h)}$$
$$E_h^* = \frac{\left(N \mu_h (M b^2 \beta_h \rho_m \beta_m - \mu_m A)\right)}{\beta_m \rho_h b(M b \beta_h \rho_m C + A - B)}$$
$$I_h^* = \frac{\left(N \mu_h (M b^2 \beta_h \rho_m \beta_m - \mu_m A)\right)}{\beta_m b(\mu_h + 1)(M b \beta_h \rho_m C + A - B)}$$
$$R_h^* = \frac{\left(N(1-\gamma)(M b^2 \beta_h \rho_m \rho_h \beta_m - A)\right)}{\beta_m b(\mu_h + 1)(M b \beta_h \rho_m C + A - B)}$$
$$E_m^* = \frac{\left(\mu_h (M b^2 \beta_h \rho_m \rho_h \beta_m - A)\right)}{\rho_m C(D + b^2 \beta_h \mu_h \rho_h \beta_m)}$$
$$I_m^* = \frac{\left(\mu_h (M b^2 \beta_h \rho_m \rho_h \beta_m - A)\right)}{\mu_m C(D + b^2 \beta_h \mu_h \rho_h \beta_m)}$$
$$S_m^* = \frac{(\mu_h + 1)(M b \beta_h \rho_m C + A - B)}{\beta_h \rho_m b E}$$

With,

$$A = N\mu_m(\rho_m \mu_h^2 + \rho_m \mu_h \rho_h + \mu_m \mu_h^2 + \mu_m \mu_h \rho_h + \rho_m \mu_h + \rho_m \rho_h + \mu_m \mu_h + \mu_m \rho_h)$$
$$B = N\mu_m(\rho_m \mu_h + \rho_m \rho_h + \mu_m \mu_h + \mu_m \rho_h)$$
$$C = \mu_h + \rho_h$$
$$D = \beta_h b \mu_m(\mu_h^2 + \mu_h \rho_h + \mu_h + \rho_h)$$
$$E = \beta_h b \mu_m(b \mu_h \rho_h \beta_m + \mu_m \mu_h^2 + \mu_m \mu_h \rho_h + \mu_m \mu_h + \mu_m \rho_h)$$

Basic Reproduction Number

In analyzing disease transmission, the basic reproduction number ($\Re_0$) is crucial. The average number of new incidents brought on by those who can spread the illness is known as $\Re_0$. If $\Re_0 > 1$, it signifies that one diseased person can spread the disease to more than one secondary infection. In this case, the disease-free equilibrium is unstable so that it can cause the epidemic outbreaks. But if the $\Re_0 < 1$, the disease-free equilibrium (DFE) will be locally asymptotically stable and the situation is under control. The basic reproduction number can be found analytically using Next Generation Matrix (NGM). The $\Re_0$ can be computed by considering the bellow generation matrix $\mathcal{F}V^{-1}$,

where $\mathcal{F}(\mathcal{E}^*) = \begin{bmatrix}\mathcal{F}_1\\ \mathcal{F}_2\\ \mathcal{F}_3\\ \mathcal{F}_4\end{bmatrix}\begin{bmatrix}\frac{b\beta_h S_h I_m}{N}\\ 0\\ \frac{b\beta_m S_m I_h}{N}\\ 0\end{bmatrix}$

and $V = \begin{bmatrix}V_1\\ V_2\\ V_3\\ V_4\end{bmatrix} = \begin{bmatrix}-(\mu_h+\rho_h)E_h\\ \rho_h E_h - (\gamma_h+\mu_h)I_h - (1-\gamma_h)I_h\\ -(\rho_m+\mu_m)E_m\\ \rho_m E_m - \mu_m I_m\end{bmatrix}$

Therefore, the spectral radius of FV^{-1} is

$$\Re_0 = \left(\sqrt{\frac{\mu_m\beta_m\rho_h NM\beta_h b\mu_m(\mu_h^2+\mu_h\rho_h+\mu_h+\rho_h)}{N\mu_m(\rho_m\mu_h^2+\rho_m\mu_h\rho_h+\mu_m\mu_h^2+\mu_m\mu_h\rho_h+\rho_m\mu_h+\rho_m\rho_h+\mu_m\mu_h+\mu_m\rho_h)}}\right)$$

Existence of Equilibrium Point

In this section we will establish the existence condition of the equilibrium states [10]. The equilibrium points of the model (1) are as follow:

$\mathcal{E}_0(S_{h_0}, E_{h_0}, I_{h_0}, R_{h_0}, S_{m_0}, E_{m_0}, I_{m_0}) = \left(N, 0,0,0, \frac{M}{\mu_m}, 0,0\right)$ do always exist

$\mathcal{E}^*(S_h^*, E_h^*, I_h^*, R_h^*, S_m^*, E_m^*, I_m^*)$, The existence condition for $\mathcal{E}^*$ if,

$$Mb^2\beta_h\rho_m\rho_h\beta_m - \mu_m A > 0$$

$$Mb^2\beta_h\rho_m\rho_h\beta_m - \frac{\mu_m^2\beta_m\rho_h NMD}{\Re_0{}^2} > 0$$

$$Mb^2\beta_h\rho_m\rho_h\beta_m > \frac{\mu_m^2\beta_m\rho_h NMD}{\Re_0{}^2}$$

$$\Re_0{}^2 > \frac{\mu_m^2\beta_m\rho_h NMD}{Mb^2\beta_h\rho_m\rho_h\beta_m}$$

$$\Re_0{}^2 > \frac{\mu_m^2 ND}{b^2\beta_h\rho_m}$$

$$\Re_0 > \frac{\mu_m^2 ND}{b^2\beta_h\rho_m}$$

$$\Re_0 > 1$$

The endemic equilibrium point is exist if $\Re_0 > 1$

Stability of Equilibrium Point

Theorem 5.1 Let $\Re_0 = \left(\sqrt{\frac{\mu_m\beta_m\rho_h N M\beta_h b\mu_m(\mu_h^2+\mu_h\rho_h+\mu_h+\rho_h)}{N\mu_m(\rho_m\mu_h^2+\rho_m\mu_h\rho_h+\mu_m\mu_h^2+\mu_m\mu_h\rho_h+\rho_m\mu_h+\rho_m\rho_h+\mu_m\mu_h+\mu_m\rho_h)}}\right)$, The endemic equilibrium state $\mathcal{E}^*(S_h^*, E_h^*, I_h^*, R_h^*, S_m^*, E_m^*, I_m^*)$ is stable if $\Re_0 > 1$ and unstable if $\Re_0 < 1$

1. Proof

To prove the stability of endemic point is stable if $\Re_0 > 1$, we use the characteristic polynomial. First step we do the liearization of system (2.1).

$$\frac{d}{dt}\begin{bmatrix} S_h \\ E_h \\ I_h \\ R_h \\ S_m \\ E_m \\ I_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial S_h} & \frac{\partial f_1}{\partial E_h} & \frac{\partial f_1}{\partial I_h} & \frac{\partial f_1}{\partial R_h} & \frac{\partial f_1}{\partial S_m} & \frac{\partial f_1}{\partial E_m} & \frac{\partial f_1}{\partial I_m} \\ \frac{\partial f_2}{\partial S_h} & \frac{\partial f_2}{\partial E_h} & \frac{\partial f_2}{\partial I_h} & \frac{\partial f_2}{\partial R_h} & \frac{\partial f_2}{\partial S_m} & \frac{\partial f_2}{\partial E_m} & \frac{\partial f_2}{\partial I_m} \\ \frac{\partial f_3}{\partial S_h} & \frac{\partial f_3}{\partial E_h} & \frac{\partial f_3}{\partial I_h} & \frac{\partial f_3}{\partial R_h} & \frac{\partial f_3}{\partial S_m} & \frac{\partial f_3}{\partial E_m} & \frac{\partial f_3}{\partial I_m} \\ \frac{\partial f_4}{\partial S_h} & \frac{\partial f_4}{\partial E_h} & \frac{\partial f_4}{\partial I_h} & \frac{\partial f_4}{\partial R_h} & \frac{\partial f_4}{\partial S_m} & \frac{\partial f_4}{\partial E_m} & \frac{\partial f_4}{\partial I_m} \\ \frac{\partial f_5}{\partial S_h} & \frac{\partial f_5}{\partial E_h} & \frac{\partial f_5}{\partial I_h} & \frac{\partial f_5}{\partial R_h} & \frac{\partial f_5}{\partial S_m} & \frac{\partial f_5}{\partial E_m} & \frac{\partial f_5}{\partial I_m} \\ \frac{\partial f_6}{\partial S_h} & \frac{\partial f_6}{\partial E_h} & \frac{\partial f_6}{\partial I_h} & \frac{\partial f_6}{\partial R_h} & \frac{\partial f_6}{\partial S_m} & \frac{\partial f_6}{\partial E_m} & \frac{\partial f_6}{\partial I_m} \\ \frac{\partial f_7}{\partial S_h} & \frac{\partial f_7}{\partial E_h} & \frac{\partial f_7}{\partial I_h} & \frac{\partial f_7}{\partial R_h} & \frac{\partial f_7}{\partial S_m} & \frac{\partial f_7}{\partial E_m} & \frac{\partial f_7}{\partial I_m} \end{bmatrix} (S_h^*, E_h^*, I_h^*, R_h^*, S_m^*, E_m^*, I_m^*) , \begin{bmatrix} \overline{S_h} \\ \overline{E_h} \\ \overline{I_h} \\ \overline{R_h} \\ \overline{S_m} \\ \overline{E_m} \\ \overline{I_m} \end{bmatrix}$$

Then with maple application we obtain :

$$\frac{d}{dt}\begin{bmatrix} S_h \\ E_h \\ I_h \\ R_h \\ S_m \\ E_m \\ I_m \end{bmatrix} = \begin{bmatrix} -\frac{b\beta_h I_m}{N} - \mu_h & 0 & 0 & 0 & 0 & 0 & -\frac{b\beta_h S_h}{N} \\ \frac{b\beta_h I_m}{N} & -\mu_h - \rho_h & 0 & 0 & 0 & 0 & \frac{b\beta_h S_h}{N} \\ 0 & \rho_h & -\mu_h - 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - \gamma_h & -\mu_h & 0 & 0 & 0 \\ 0 & 0 & -\frac{b\beta_m S_m}{N} & 0 & -\frac{b\beta_m I_h}{N} - \mu_m & 0 & 0 \\ 0 & 0 & \frac{b\beta_m S_m}{N} & 0 & \frac{b\beta_m I_h}{N} & -\rho_m - \mu_m & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_m & -\mu_m \end{bmatrix} , \begin{bmatrix} \overline{S_h} \\ \overline{E_h} \\ \overline{I_h} \\ \overline{R_h} \\ \overline{S_m} \\ \overline{E_m} \\ \overline{I_m} \end{bmatrix} \quad (2)$$

Then substitute the endemic point to (2) and we get Jacobian matrix at $\mathcal{E}^*(S_h^*, E_h^*, I_h^*, R_h^*, S_m^*, E_m^*, I_m^*)$ as follows:

$$J(\mathcal{E}^*) = \begin{bmatrix} b_1 & 0 & 0 & 0 & 0 & 0 & b_2 \\ b_3 & b_4 & 0 & 0 & 0 & 0 & b_5 \\ 0 & b_6 & b_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_8 & b_9 & 0 & 0 & 0 \\ 0 & 0 & b_{10} & 0 & b_{11} & 0 & 0 \\ 0 & 0 & b_{12} & 0 & b_{13} & b_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{15} & b_{16} \end{bmatrix}$$

With,

$$b_1 = -\frac{b\beta_h\left(\mu_h(Mb^2\beta_h\rho_m\rho_h\beta_m - A)\right)}{N\mu_m C(D + b^2\beta_h\mu_h\rho_h\beta_m)} - \mu_h$$

$$b_2 = -\frac{b\beta_h N^2\ \mu_m EC}{N\beta_m\rho_h b(Mb\beta_h\rho_m + N\rho_m\mu_m\mu_h + N\mu_m^2\mu_h)}$$
$$b_3 = \frac{b\beta_h\big(\mu_h(Mb^2\beta_h\rho_m\rho_h\beta_m - A)\big)}{N\mu_m C(D + b^2\beta_h\mu_h\rho_h\beta_m)}$$
$$b_4 = -\mu_h - \rho_h$$
$$b_5 = \frac{b\beta_h N^2\ \mu_m EC}{N\beta_m\rho_h b(Mb\beta_h\rho_m + N\rho_m\mu_m\mu_h + N\mu_m^2\mu_h)}$$
$$b_6 = \rho_h$$
$$b_7 = -\mu_h - 1$$
$$b_8 = 1 - \gamma_h$$
$$b_9 = -\mu_h$$
$$b_{10} = -\frac{b\beta_m N^2\ \mu_m EC}{N\beta_m\rho_h b(Mb\beta_h\rho_m + N\rho_m\mu_m\mu_h + N\mu_m^2\mu_h)}$$
$$b_{11} = -\frac{b\beta_m\big(N\mu_h(Mb^2\beta_h\rho_m\beta_m - \mu_m A)\big)}{N\beta_m b(\mu_h + 1)(Mb\beta_h\rho_m C + A - B)} - \mu_m$$
$$b_{12} = \frac{b\beta_m N^2\ \mu_m EC}{N\beta_m\rho_h b(Mb\beta_h\rho_m + N\rho_m\mu_m\mu_h + N\mu_m^2\mu_h)}$$
$$b_{13} = \frac{b\beta_m\big(N\mu_h(Mb^2\beta_h\rho_m\beta_m - \mu_m A)\big)}{N\beta_m b(\mu_h + 1)(Mb\beta_h\rho_m C + A - B)}$$
$$b_{14} = -\rho_m - \mu_m$$
$$b_{15} = \rho_m$$
$$b_{16} - \mu_m$$

From the Jacobian matrix at $\mathcal{E}^*(S_h^*, E_h^*, I_h^*, R_h^*, S_m^*, E_m^*, I_m^*)$, we get the polynomial characteristic as follows:

$$Pol(X^7) = c_0x^7 + c_1x^6 + c_2x^5 + c_3x^4 + c_4x^3 + c_5x^2 + c_6x + c_7 \tag{3}$$

With,

$$c_0 = 1$$
$$c_1 = -b_{16} - b_{14} - b_{11} - b_9 - b_7 - b_4 - b_1$$
$$\begin{aligned} c_2 = {} & b_1b_{11} + b_1b_{14} + b_1b_{16} + b_1b_4 + b_1b_7 + b_1b_9 + b_{11}b_{14} + b_{11}b_{16} + b_{11}b_4 \\ & + b_{11}B_7 + b_{11}b_9 + b_{14}b_{16} + b_{14}b_4 + b_{15}b_7 + b_{14}b_9 + b_{16}b_4 + b_{16}b_7 \\ & + b_{16}b_9 + b_4b_7 + b_4b_9 + b_7b_9 \end{aligned}$$
$$\begin{aligned} c_3 = {} & -b_1b_{11}b_{14} - (b_1b_{11} + b_1b_{14} + b_{11}b_{14})(b_{16} + b_4 + b_7 + b_9) - (b_1b_{16} + b_{11}b_{16} \\ & + b_{14}b_{16})(b_4 + b_7 + b_9) - (b_1b_4 + b_{11}b_4 + b_{14}b_4 + b_{16}b_4)(b_7 + b_9) \\ & - (b_1 + b_{11} + b_{14} + b_{16} + b_4)b_7b_9 \end{aligned}$$
$$\begin{aligned} c_4 = {} & b_1b_{11}b_{14}b_{16} + (b_1b_{11}b_{14})(b_{16} + b_4 + b_7 + b_9) + (b_1b_{11}b_{16} + b_1b_{14}b_{16} \\ & + b_{11}b_{14}b_{16})(b_4 + b_7 + b_9) + (b_1b_{11}b_4 + b_4 + b_1b_{14}b_4 + b_{11}b_{14}b_4 \\ & + b_{11}b_{16}b_4 + b_1b_{16}b_4 + b_14b_{16}b_4)(b_7 + b_9) + (b_1b_{16} + b_1b_{14} \\ & + b_{11}b_{14} + b_{11}b_{16} + b_{14}b_{16} + b_{11}b_4 + b_{14}b_4 + b_{16}b_4 + b_1b_4)b_7b_9 \\ & - b_{12}b_{15}b_5b_6 \end{aligned}$$

$$
\begin{aligned}
c_5 = -(&b_1 b_{11} b_{14} b_{16}(b_4 + b_7 + b_9)(b_1 b_{11} b_{14} + b_1 b_{11} b_4 + b_1 b_{14} b_{16} \\
&+ b_1 b_{14} b_4 + b_1 b_{16} b_4 + b_{11} b_{14} b_4 + b_{11} b_{16} b_4 + b_{16} b_{14} b_4) b_7 b_9 \\
&- (b_1 b_{11} b_{14} b_4 + b_1 b_{11} b_{16} b_4 + b_1 b_{14} b_{16} b_4 + b_{11} b_{14} b_{16} b_4)(b_7 + b_9) \\
&+ b_1 b_{12} b_{15} b_5 b_6 - b_{10} b_{13} b_{15} b_5 b_6 + b_{11} b_{12} b_{15} b_5 b_6 - b_2 b_{12} b_{15} b_3 b_6 \\
&+ b_5 b_{12} b_{15} b_9 b_6 \\
c_6 = (&b_1 b_{10} b_{15} b_{13} + b_1 b_{11} b_{15} b_{12})(b_5 b_6) \\
&+ b_1 b_{11} b_{14} b_{16} b_4 (b_7 + b_9) + (b_1 b_{11} b_{14} b_4 + b_1 b_{11} b_{16} b_4 + b_1 b_{14} b_{16} b_4 \\
&+ b_{11} b_{14} b_{16} b_4) b_7 b_9 - b_1 b_{12} b_{15} b_9 b_5 b_6 b_2 b_{10} b_{15} b_{13} b_3 b_6 \\
&- b_{11} b_{12} b_{15} b_9 b_5 b_6 + b_{12} b_2 b_{15} b_3 b_9 b_6 \\
c_7 = -b_9(&b_1 b_{10} b_{15} b_{13} b_5 b_6 - b_1 b_{11} b_{15} b_{12} b_5 b_6 + b_1 b_{11} b_{14} b_{16} b_4 b_7 \\
&- b_{10} b_{15} b_{13} b_3 b_6 + b_{11} b_{15} b_{12} b_3 b_6
\end{aligned}
$$

From (3), we use Routh Hurwitz method [11] to analyze the local stability of endemic point. The Hurwitz Matrix of (3) as follows:

$$
H = \begin{pmatrix} c_1 & c_0 & 0 & 0 \\ c_3 & c_2 & c_1 & c_0 \\ c_5 & c_4 & c_3 & c_2 \\ c_7 & c_6 & c_5 & c_4 \end{pmatrix} \quad (4)
$$

The first determinant of (4) is,

$$
\begin{aligned}
&|H_1| = |c_1| \\
&= -b_{16} - b_{14} - b_{11} - b_9 - b_7 - b_4 - b_1
\end{aligned}
$$

Because $b_{16}, b_{14}, b_{11}, b_9, b_7, b_4, b_1 < 0$ so that $|H_1|$is positive

The second determinant of Hurwitz Matrix is,

$$
\begin{aligned}
|H_2| &= \left|\begin{pmatrix} c_1 & c_0 \\ c_3 & c_2 \end{pmatrix}\right| \\
&= -k - (-k - \mu_m) - \left(-\frac{\mu_h(Mb^2\beta_h\rho_m\rho_h\beta_m - \mu_m A)}{C(Mb\beta_h\rho_m + N\rho_m\mu_m\mu_h + N\mu_m^2\mu_h)(\mu_h+1)} - \mu_m\right) - (-\mu) - (-\mu - \\
&1) - (-\rho - \mu) - \left(-\frac{\mu_h(Mb^2\beta_h\rho_m\rho_h\beta_m - \mu_m A)}{N\mu_m C\left(b\mu_h\rho_m\beta_m + \frac{D}{\beta_h b}\right)(\mu_h+1)} - \mu\right)
\end{aligned}
$$

$$
\begin{aligned}
= 2\mu_m &+ \frac{\mu_h(Mb^2\beta_h\rho_m\rho_h\beta_m - \mu_m A)}{C(Mb\beta_h\rho_m + N\rho_m\mu_m\mu_h + N\mu_m^2\mu_h)(\mu_h + 1)} + 4\mu_h + 1 + \rho_h \\
&+ \frac{\mu_h(Mb^2\beta_h\rho_m\rho_h\beta_m - \mu_m A)}{N\mu_m C\left(b\mu_h\rho_m\beta_m + \frac{D}{\beta_h b}\right)(\mu_h + 1)}
\end{aligned}
$$

The Hurwitz Matrix is positive if,

$$Mb^2\beta_h\rho_m\rho_h\beta_m - \mu_m A > 0$$

$$Mb^2\beta_h\rho_m\rho_h\beta_m - \frac{\mu_m^2\beta_m\rho_h NMD}{\mathfrak{R}_0{}^2} > 0$$

$$Mb^2\beta_h\rho_m\rho_h\beta_m > \frac{\mu_m^2\beta_m\rho_h NMD}{\mathfrak{R}_0{}^2}$$

$$\mathfrak{R}_0{}^2 > \frac{\mu_m^2\beta_m\rho_h NMD}{Mb^2\beta_h\rho_m\rho_h\beta_m}$$

$$\mathfrak{R}_0{}^2 > \frac{\mu_m^2 ND}{b^2\beta_h\rho_m}$$

$$\Re_0 > \frac{\mu_m^2 ND}{b^2 \beta_h \rho_m}$$
$$\Re_0 > 1$$

The root of solutions at endemic point is negative if the determinant of Hurwitz Matrix is positive. It is evident that endemic equilibrium point is locally asymptotically stable if $\Re_0 > 1$.

Theorem 5.2 Let $\Re_0 = \left(\sqrt{\frac{\mu_m\beta_m\rho_h NM\beta_h b\mu_m(\mu_h^2+\mu_h\rho_h+\mu_h+\rho_h)}{N\mu_m(\rho_m\mu_h^2+\rho_m\mu_h\rho_h+\mu_m\mu_h^2+\mu_m\mu_h\rho_h+\rho_m\mu_h+\rho_m\rho_h+\mu_m\mu_h+\mu_m\rho_h)}}\right)$, the disease free equilibrium will be locally asymptotically stable if $\Re_0 < 1$ and unstable if $\Re_0 > 1$.

2. Proof

To prove the stability of disease-free equilibrium point is stable if $\Re_0 < 1$, we use the characteristic polynomial. First step we do the liearization of system (1).

$$\frac{d}{dt}\begin{bmatrix} S_h \\ E_h \\ I_h \\ R_h \\ S_m \\ E_m \\ I_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial S_h} & \frac{\partial f_1}{\partial E_h} & \frac{\partial f_1}{\partial I_h} & \frac{\partial f_1}{\partial R_h} & \frac{\partial f_1}{\partial S_m} & \frac{\partial f_1}{\partial E_m} & \frac{\partial f_1}{\partial I_m} \\ \frac{\partial f_2}{\partial S_h} & \frac{\partial f_2}{\partial E_h} & \frac{\partial f_2}{\partial I_h} & \frac{\partial f_2}{\partial R_h} & \frac{\partial f_2}{\partial S_m} & \frac{\partial f_2}{\partial E_m} & \frac{\partial f_2}{\partial I_m} \\ \frac{\partial f_3}{\partial S_h} & \frac{\partial f_3}{\partial E_h} & \frac{\partial f_3}{\partial I_h} & \frac{\partial f_3}{\partial R_h} & \frac{\partial f_3}{\partial S_m} & \frac{\partial f_3}{\partial E_m} & \frac{\partial f_3}{\partial I_m} \\ \frac{\partial f_4}{\partial S_h} & \frac{\partial f_4}{\partial E_h} & \frac{\partial f_4}{\partial I_h} & \frac{\partial f_4}{\partial R_h} & \frac{\partial f_4}{\partial S_m} & \frac{\partial f_4}{\partial E_m} & \frac{\partial f_4}{\partial I_m} \\ \frac{\partial f_5}{\partial S_h} & \frac{\partial f_5}{\partial E_h} & \frac{\partial f_5}{\partial I_h} & \frac{\partial f_5}{\partial R_h} & \frac{\partial f_5}{\partial S_m} & \frac{\partial f_5}{\partial E_m} & \frac{\partial f_5}{\partial I_m} \\ \frac{\partial f_6}{\partial S_h} & \frac{\partial f_6}{\partial E_h} & \frac{\partial f_6}{\partial I_h} & \frac{\partial f_6}{\partial R_h} & \frac{\partial f_6}{\partial S_m} & \frac{\partial f_6}{\partial E_m} & \frac{\partial f_6}{\partial I_m} \\ \frac{\partial f_7}{\partial S_h} & \frac{\partial f_7}{\partial E_h} & \frac{\partial f_7}{\partial I_h} & \frac{\partial f_7}{\partial R_h} & \frac{\partial f_7}{\partial S_m} & \frac{\partial f_7}{\partial E_m} & \frac{\partial f_7}{\partial I_m} \end{bmatrix} \left(S_{h_0}, E_{h_0}, I_{h_0}, R_{h_0}, S_{m_0}, E_{m_0}, I_{m_0}\right), \begin{bmatrix} \overline{S_h} \\ \overline{E_h} \\ \overline{I_h} \\ \overline{R_h} \\ \overline{S_m} \\ \overline{E_m} \\ \overline{I_m} \end{bmatrix}$$

Then with maple application we obtain :

$$\frac{d}{dt}\begin{bmatrix} S_h \\ E_h \\ I_h \\ R_h \\ S_m \\ E_m \\ I_m \end{bmatrix} = \begin{bmatrix} -\frac{b\beta_h I_m}{N} - \mu_h & 0 & 0 & 0 & 0 & 0 & -\frac{b\beta_h S_h}{N} \\ \frac{b\beta_h I_m}{N} & -\mu_h - \rho_h & 0 & 0 & 0 & 0 & \frac{b\beta_h S_h}{N} \\ 0 & \rho_h & -\mu_h - 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - \gamma_h & -\mu_h & 0 & 0 & 0 \\ 0 & 0 & -\frac{b\beta_m S_m}{N} & 0 & -\frac{b\beta_m I_h}{N} - \mu_m & 0 & 0 \\ 0 & 0 & \frac{b\beta_m S_m}{N} & 0 & \frac{b\beta_m I_h}{N} & -\rho_m - \mu_m & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_m & -\mu_m \end{bmatrix}, \begin{bmatrix} \overline{S_h} \\ \overline{E_h} \\ \overline{I_h} \\ \overline{R_h} \\ \overline{S_m} \\ \overline{E_m} \\ \overline{I_m} \end{bmatrix} \quad (5)$$

Then substitute the non-endemic point to (5) and we get Jacobian matrix at $\mathcal{E}_0(S_{h_0}, E_{h_0}, I_{h_0}, R_{h_0}, S_{m_0}, E_{m_0}, I_{m_0}) = \left(N, 0,0,0, \frac{M}{\mu_m}, 0,0\right)$ as follows :

$$J(\mathcal{E}_0) = \begin{pmatrix} -\mu_h & 0 & 0 & 0 & 0 & 0 & -b\beta_h \\ 0 & -\mu_h-\rho_h & 0 & 0 & 0 & 0 & b\beta_h \\ 0 & \rho_h & -\mu_h-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\gamma_h & -\mu_h & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{b\beta_m M}{\mu_m N} & 0 & -\mu_m & 0 & 0 \\ 0 & 0 & \dfrac{b\beta_m M}{\mu_m N} & 0 & \dfrac{b\beta_m I_h}{N} & -\rho_m-\mu_m & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_m & -\mu_m \end{pmatrix} \quad (6)$$

From the Jacobian matrix (6) we get the characteristic polynomial as follows

$$\begin{aligned} P(X^6) = -\frac{1}{NM}(\mu_h+x)^2(\mu_m+x)(&Mb^2\beta_h\rho_m\rho_h\beta_m - N\rho_m\mu_m^2\mu_h^2 - N\rho_m\mu_m^2\mu_h\rho_m - 2N\rho_h\mu_m^2\mu_h x \\ &- N\rho_m\mu_m^2\rho_m x - N\rho_m\mu_m^2 x^2 - N\rho_m\mu_m\mu_h^2 x - N\rho_m\mu_m\mu_h\rho_m x - 2N\rho_h\mu_m^2\mu_h x^2 \\ &- N\rho_m\mu_m\rho_m x^2 - N\rho_m\mu_m x^3 - N\mu_m^3\mu_h^2 - N\mu_m^3\mu_h\rho_h - 2N\mu_m^3\mu_h x - N\mu_m^3\rho_h x - N\mu_m^3 x^2 \\ &- 2N\mu_m^2\mu_h^2 x - 2N\mu_m^2\mu_h\rho_h x - 4Nm^2\mu_h x^2 - 2N\mu_m^2\rho_h x^2 - 2N\mu_h^2\, x^3 - N\mu_m\mu_h^2 x^2 \\ &- N\mu_m\mu_h\rho_h x^2 - 2N\mu_m\mu_h x^3 - N\mu_m\rho_h x^3 - N\mu_m x^4 - N\rho_m\mu_m^2\mu_h - N\rho_m\mu_m^2\rho_h \\ &- N\rho_m\mu_m^2 x - N\rho_m\mu_m\mu_h x - N\rho_m\mu_m x^2 - N\mu_m^3\mu_h - N\mu_m^3\rho_h - N\mu_m^3 x - 2N\mu_m^2\mu_h x \\ &- 2N\mu_m^2\rho_h x - 2N\mu_m^2 x^2 - N\mu_m\mu_h^2 x^2 - N\mu_m\rho_h x^2 - N\mu_m x^3 \qquad (7) \end{aligned}$$

The system (1) is locally asymptotically stable at disease free equilibrium point ($\mathcal{E}_0$) if the root of solutions from characteristics polynomial (7) is negative. From (7), we get two roots of solution that have negative values as follows:

$$x_1 = -\mu_h$$
$$x_2 = -\mu_m$$

Then to know another solutions, we use the characteristics of this polynomial :

$$P(X^6) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

With,

$$\begin{aligned} a_0 &= 1 \\ a_1 &= \rho_m + 2\mu_m + \rho_h + 1 \\ a_2 &= \rho_m\mu_m + 2\rho_m\mu_h + \rho_m\rho_h + \mu_m^2 + 4\mu_m\mu_h + 2\mu_m\rho_h + \mu_h^2 + \mu_h\rho_h + \rho_m + 2\mu_m \\ &\quad + \mu_h + \rho_h \\ a_3 &= 2\rho_m\mu_m\mu_h + \rho_m\mu_m\rho_h + \rho_m\mu_h^2 + +\rho_m\mu_h\rho_h + 2\mu_m^2\mu_h + \mu_m^2\rho_h + 2\mu_m\mu_h^2 \\ &\quad + 2\mu_m\mu_h\rho_h + \rho_m\mu_h + \rho_m\rho_h + \mu_m^2 + 2\mu_m\mu_h + 2\mu_m\rho_h \\ a_4 &= -\frac{1}{NM}(Mb^2\beta_h\rho_m\rho_h\beta_m - \rho_m\mu_h^2 - \rho_m\mu_h\rho_h - \mu_m\mu_h^2 - \mu_m\mu_h\rho_h - \rho_m\mu_h \\ &\quad - \rho_m\rho_h - \mu_m\mu_h - \mu_m\rho_h) \end{aligned}$$

Then the characteristics of solutions (7) as follows:

$$x_3 + x_4 + x_5 + x_6 = -\frac{a_1}{a_0} = -\frac{\rho_m + 2\mu_m + \rho_h + 1}{1}$$

$$x_3.x_4 + x_3.x_5 + x_3.x_6 + x_4.x_5 + x_4.x_6 + x_5.x_6 = \frac{a_2}{a_0}$$

$$= \frac{\rho_m\mu_m + 2\rho_m\mu_h + \rho_m\rho_h + \mu_m^2 + 4\mu_m\mu_h + 2\mu_m\rho_h + \mu_h^2 + \mu_h\rho_h + \rho_m + 2\mu_m + \mu_h + \rho_h}{1}$$

$$x_3.x_4.x_5 + x_3.x_4.x_6 + x_4.x_5.x_6 = -\frac{a_3}{a_0}$$

$$= -\frac{2\rho_m\mu_m\mu_h + \rho_m\mu_m\rho_h + \rho_m\mu_h^2 + +\rho_m\mu_h\rho_h + 2\mu_m^2\mu_h + \mu_m^2\rho_h + 2\mu_m\mu_h^2 + 2\mu_m\mu_h\rho_h + \rho_m\mu_h + \rho_m\rho_h + \mu_m^2 + 2\mu_m\mu_h + 2\mu_m\rho_h}{1}$$

$$x_3.x_4.x_5.x_6 = \frac{a_4}{a_0}$$

$$= -\frac{1}{NM}(Mb^2\beta_h\rho_m\rho_h\beta_m - \rho_m\mu_h^2 - \rho_m\mu_h\rho_h - \mu_m\mu_h^2 - \mu_m\mu_h\rho_h - \rho_m\mu_h - \rho_m\rho_h - \mu_m\mu_h - \mu_m\rho_h)$$

The values from root of solutions (7) is negative if,

$$-\frac{1}{NM}(Mb^2\beta_h\rho_m\rho_h\beta_m - \rho_m\mu_h^2 - \rho_m\mu_h\rho_h - \mu_m\mu_h^2 - \mu_m\mu_h\rho_h - \rho_m\mu_h - \rho_m\rho_h - \mu_m\mu_h - \mu_m\rho_h) > 0$$

$$-\frac{1}{NM}\left(Mb^2\beta_h\rho_m\rho_h\beta_m - \frac{A}{N\mu_m}\right) > 0$$

$$-\frac{1}{NM}\left(Mb^2\beta_h\rho_m\rho_h\beta_m - \frac{\frac{\mu_m\beta_m\rho_h NMD}{\Re_0{}^2}}{N\mu_m}\right) > 0$$

$$Mb^2\beta_h\rho_m\rho_h\beta_m < \frac{\mu_m\beta_m\rho_h NMD}{N\mu_m\,\Re_0{}^2}$$

$$Mb^2\beta_h\rho_m\rho_h\beta_m\,\Re_0{}^2 < \beta_m\rho_h MD$$

$$\Re_0{}^2 < \frac{\beta_m\rho_h MD}{Mb^2\beta_h\rho_m\rho_h\beta_m}$$

$$\Re_0{}^2 < \frac{D}{b^2\beta_h\rho_m}$$

$$\Re_0 < \sqrt{\frac{D}{b^2\beta_h\rho_m}}$$

Because $0 \le D \le 1$ and $b \ge 1$ then,

$$\Re_0 < 1$$

It is evident that disease free equilibrium point is locally asymptotically stable if $\Re_0 < 1$. Furthermore, in this case describe that the spread of disease is under control, so the average of every infected mosquitos transmit less disease than one other human.

Results and Discussion

Using Maple software version “Maple 15” and data from reference, we simulated the model (1). The initial conditions are S_h (0)=3.000.000 individual, E_h (0)=200000 individual, I_h (0)=10000 individual, R_h (0)=9000 individual, D_h (0)=2000 individual, S_m (0)=5000 individual, E_m (0)=4000 individual, I_m (0)=3500 and the values of parameters are given in the Table 2.

In Figure 2, by taking the above parameters, dengue transmission simulation model is obtained with a value of $\boldsymbol{\Re_0 = 5,173 > 1}$. According to this picture, the initial condition of highly susceptible humanhas decreased because of interaction with infected mosquito, so that the susceptible human become the exposed human and then, as time increased, the exposed human become the infected human. From this simulation, the infected human become deceased more than the recovered. The deceased human increased and stable over time. This indicates that there is infected mosquito that transmit the diseases to more than one other individual human in the given period. Therefore, there is a large number of infected mosquitos over time that can lead to an outbreak in a very short time. So that the endemic equilibrium point is locally asymptotically stable. From this, we know that the spread of dengue fever in stable conditions is still endemic for a long period of time, so for further research it is necessary to modelling of control strategies to reduce the spread of dengue fever.

Table 2. Parameter Values

Notation	Value	Unit	References
μ_h	0.012	day-1	[8]
β_h	0.1	day-1	[8]
ρ_h	0.3	day-1	[8]
γ_h	1.5	day-1	[8]
μ_m	0.003	day-1	[8]
β_m	0.001	day-1	[8]
ρ_m	2.1428	day-1	[8]
b	1000	unit	[8]

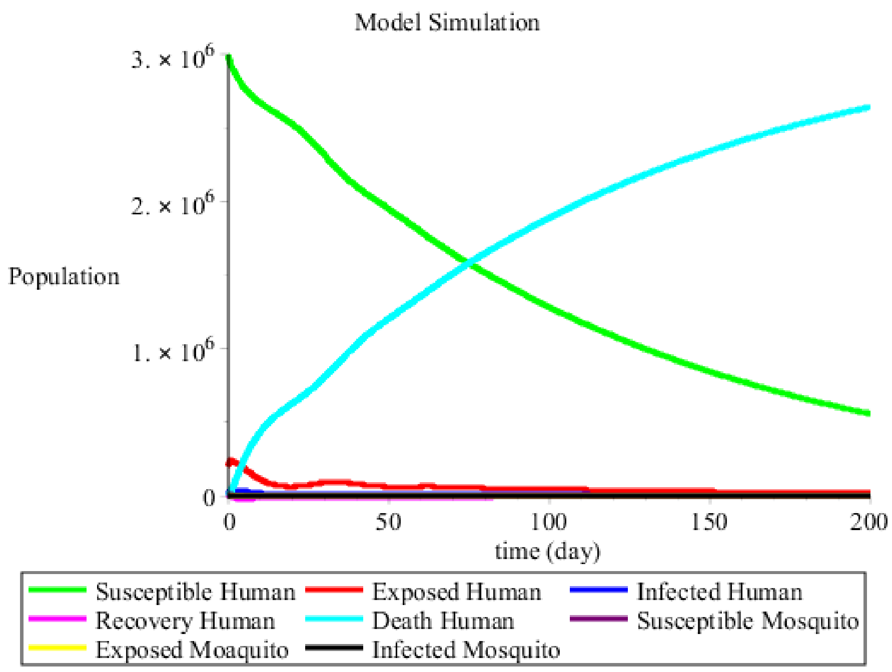

Figure 2. Dengue Transmission Model

Conclusion

In this study, we introduced the SEIRD-SEI model to analyze dengue transmission dynamics. This model incorporates eight subpopulations: susceptible humans, exposed humans, infected humans, recovered humans, deceased individuals, susceptible mosquitoes, exposed mosquitoes, and infected mosquitoes. Theoretical analysis confirms that the transmission dynamics depend on the basic reproduction number at equilibrium, which also determines the model's stability. We utilized the Routh-Hurwitz method to assess local stability. According to the analysis, 〖R〗_0>1, and the model is locally asymptotically stable at the endemic equilibrium point. The numerical simulation describes how the dengue disease propagation model behaves, then we get the value of 〖R〗_0=5,173, indicating that the dengue transmission model is locally asymptotically stable at the endemic equilibrium point.

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